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CERTIFIED INVESTMENT AND FINANCIAL ANALYSTS (CIFA)

SECTION ONE

FINANCIAL MATHEMATICS

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CHAPTERS

CHAPTER ONE

INTRODUCTION TO FINANCIAL MATHEMATICS

Nature of financial decision

Financial decisions are those made by financial managers of a firm. It's broadly classified into two.

a) Managerial decision
b) routine decision

Managerial decisions

These are Decisions that require technical skills, planning and expertise of a financial manager.

It's classified into four:

1) Financing decision

It involves looking for finance to acquire assets of the firm and may include:

- Issue of ordinary shares
- Long term loan
- Preference shares

2) Investment decision

It's the responsibility of a financial manager to determine whether acquired funds should be invested in order to generate revenue. Financial manager must do a proper appraisal of any investment that may be undertaken.

3) Dividend decision

Dividends are part of the earnings distributed to ordinary shareholders for their investment in the company. Financial manager has to consider the following:

- How much to pay
- When to pay
- How to pay i.e. cash or bonus issue
- Why to pay

4) **Liquidity/working capital management decision**

Liquidity is the ability of a company to meet its short-term financial obligation when they fall due.

It's the financial managers‟ role to ensure that the company has maintained the required liquidity and to avoid instances of insolvency.

It's also his role to manage the cash position of the firm, inventory position and the amount of receivables in the company.

Routine decisions

They are supportive to managerial decisions. They require no expertise in executing them. They are normally delegated to junior staff in finance department. They include:

- issue of cash receipts
- safeguarding the cash balance
- safe custody of important finance document (filing)
- implementation of control system

Financial asset

A **financial asset** is an intangible asset that derives value because of a contractual claim. Examples include bank deposits, bonds, and stocks. Financial assets are usually more liquid than tangible assets, such as land or real estate, and are traded on financial markets. According to the International Financial Reporting Standards (IFRS), a financial asset is defined as one of the following:

- Cash or cash equivalent;
- Equity instruments of another entity;
- Contractual right to receive cash or another financial asset from another entity or to exchange financial assets or financial liabilities with another entity under conditions that are potentially favourable to the entity;
- Contract that will or may be settled in the entity's own equity instruments and is either a non-derivative for which the entity is or may be obliged to receive a variable number of the entity's own equity instruments, or a derivative that will or may be settled other than by exchange of a fixed amount of cash or another financial asset for a fixed number of the entity's own equity instruments.

Risk and return

Risk

It refers to deviation or variations of the actual outcome from the expected. It‟s the possibility of things happening than they are expected.
Risk can be measured for either a single or a gap of project (portfolio-collection of securities)

Return

It is the anticipated gain or earnings on any investment. These investments returns could be positive or negative outcomes.

'Risk-Return Tradeoff'

It's the principle that the potential return rises with an increase in risk. Low levels of uncertainty (low-risk) are associated with low potential returns, whereas high levels of uncertainty (high-risk) are associated with high potential returns. According to the risk-return tradeoff, invested money can render higher profits only if it is subject to the possibility of being lost.

Because of the risk-return tradeoff, you must be aware of your personal risk tolerance when choosing investments for your portfolio. Taking on some risk is the price of achieving returns; therefore, if you want to make money, you can't cut out all risk. The goal instead is to find an appropriate balance - one that generates some profit, but still allows you to sleep at night.

Optimization decisions

These are decisions that maximize returns and minimize risks to an investor.

Relationship between finance and other disciplines

Relationships to Economics:

There are two important linkages between economics and finance. The macroeconomic environment defines the setting within which a firm operates and the micro-economic theory provides the conceptual under pinning for the tools of financial decision making.

Key macro-economic factors like the growth rate of the economy, the domestic savings rate, the role of the government in economic affairs, the tax environment, the nature of external economic relationships the availability of funds to the corporate sector, the rate of inflation, the real rate of interests, and the terms on which the firm can raise finances define the environment in which the firm operates. No finance manager can afford to ignore the key developments in the macro economic sphere and the impact of the same on the firm.

While an understanding of the macro economic developments sensitizes the finance manager to the opportunities and threats in the environment, a firm grounding in micro economic principles sharpens his analysis of decision alternatives. Finance, in essence, is applied micro economics.

For example the principle of marginal analysis – a key principle of micro economics according to with a decision should be guided by a comparison of incremental benefits and cost is applicable to a number of managerial decisions in finance.

To sum up, a basic knowledge of macro economics is necessary for understanding the environment in which the firm operates and a good grasp of micro economic principles is helpful in sharpening the tools of financial decision making.

Relationship to Accounting:

The finance and accounting functions are closely related and almost invariably fall within the domain of the chief financial officer as shown. Given this affinity, it is not surprising that in popular perception finance and accounting are often considered indistinguishable or at least substantially over lapping. However, as a student of finance you should know how the two differ and how the two relate. The following discussion highlights the difference and relationship between the two.

Score Keeping Vs Value maximizing: Accounting is concerned with score keeping, whereas finance is aimed at value maximizing. The primary objective of accounting is to measure the performance of the performance of the firm, assess its financial condition, and determine the base for tax payment. The principal goal of financial management is to create shareholder value by investing in positive net present value projects and minimizing the cost of financing. Of course, financial decision making requires considerable inputs from accounting.

The accountant's role is to provide consistently developed and easily interested data about the firm's past, present and future operations. The financial manager uses these data, either in raw form or after certain adjustments and analyses as an important input to the decision making process.

Accrual Method Vs Cash Flow Method : The accountant prepares the accounting reports based on the accrual method which recognizes revenues when the sale occurs irrespective of whether the cash is realized immediately or not and matches expenses to sales irrespective of whether cash is paid or not. The focus of the finance manager, however, is on cash flows. He is concerned about the magnitude, timing and risk of cash flows as these are the fundamental determinants of values.

Certainty Vs Uncertainty: According deals primarily with the past, it records what has happened. Hence it is relatively more objective and certain. Finance is concerned mainly with the future. It involves decision making under imperfect information and uncertainty. Hence it is characterized by a high degree of subjectivity.

Relationship to mathematics

Financial mathematics is the application of mathematical methods to the solution of problems in finance. (Equivalent names sometimes used are financial engineering, mathematical finance, and computational finance.) It draws on tools from applied mathematics, computer science, statistics, and economic theory. Investment banks, commercial banks, hedge funds, insurance companies, corporate treasuries, and regulatory agencies apply the methods of financial mathematics to such problems as derivative securities valuation, portfolio structuring, risk management, and scenario simulation. Quantitative analysis has brought efficiency and rigor to financial markets and to the investment process and is becoming increasingly important in regulatory concerns. As the pace of financial innovation increases, the need for highly qualified people with specific training in financial mathematics intensifies.

Finance as a sub-field of economics concerns itself with the valuation of assets and financial instruments as well as the allocation of resources. Centuries of history and experience have produced fundamental theories about the way economies function and the way we value assets.

Mathematics comes into play because it allows theoreticians to model the relationships between variables and represent randomness in a manner that can lead to useful analysis. Mathematics, then, becomes a tool chest from which researchers can draw to solve problems, provide insights and make the intractable model tractable.

Mathematical finance draws from the disciplines of probability theory, statistics, scientific computing and partial differential equations to provide models and derive relationships between fundamental variables such as asset prices, market movements and interest rates. These mathematical tools allow us to draw conclusions that can be otherwise difficult to find or not immediately obvious from intuition. Especially with the aid of modern computational techniques, we can store vast quantities of data and model many variables simultaneously, leading to the ability to model quite large and complicated systems. Therefore the techniques of scientific computing, such as numerical computations, Monte Carlo simulation and optimization are an important part of financial mathematics.

Financial modeling

Financial modeling is the task of building an abstract representation (a model) of a real world financial situation.This is a mathematical model designed to represent (a simplified version of) the performance of a financial asset or portfolio of a business, project, or any other investment. Financial modeling is a general term that means different things to different users; the reference usually relates either to accounting and corporate finance applications, or to quantitative finance applications. While there has been some debate in the industry as to the nature of financial modeling—whether it is a tradecraft, such as welding, or a science—the task of financial modeling has been gaining acceptance and rigor over the years. Typically, financial modeling is understood to mean an exercise in either asset pricing or corporate finance, of a quantitative nature. In other words, financial modelling is about translating a set of hypotheses about the behavior of markets or agents into numerical predictions; for example, a firm's decisions about investments (the firm will invest 20% of assets), or investment returns (returns on "stock A" will, on average, be 10% higher than the market's returns).

Accounting

In corporate finance, investment banking, and the accounting profession *financial modeling* is largely synonymous with financial statement forecasting. This usually involves the preparation of detailed company specific models used for decision making purposes and financial analysis.

Applications include:

- Business valuation, especially discounted cash flow, but including other valuation problems
- Scenario planning and management decision making ("what is"; "what if"; "what has to be done"])

- Capital budgeting
- Cost of capital (i.e. WACC) calculations
- Financial statement analysis (including of operating- and finance leases, and R&D)
- Project finance
- Mergers and Acquisitions (i.e. estimating the future performance of combined entities)

To generalize as to the nature of these models: firstly, as they are built around financial statements, calculations and outputs are monthly, quarterly or annual; secondly, the inputs take the form of "assumptions", where the analyst *specifies* the values that will apply in each period for external / global variables (exchange rates, tax percentage, etc…; may be thought of as the model *parameters*), and for internal / company specific *variables* (wages, unit costs, etc.…). Correspondingly, both characteristics are reflected (at least implicitly) in the mathematical form of these models: firstly, the models are in discrete time; secondly, they are deterministic. For discussion of the issues that may arise, see below; for discussion as to more sophisticated approaches sometimes employed, see Corporate finance# Quantifying uncertainty.

The Financial Modeling World Championships, known as ModelOff, have been held since 2012. ModelOff is a global online financial modeling competition which culminates in a Live Finals Event for top competitors. From 2012-2014 the Live Finals were held in New York City and in this year 2015 they will be held in London

Quantitative finance

In quantitative finance, *financial modeling* entails the development of a sophisticated mathematical model.Models here deal with asset prices, market movements, portfolio returns and the like. A key distinction[] is between models of the financial situation of a large, complex firm or "quantitative financial management", models of the returns of different stocks or "quantitative asset pricing", models of the price or returns of derivative securities or "financial engineering" and models of the firm's financial decisions or "quantitative corporate finance".

Applications include:

- Option pricing
- Other derivatives, especially interest rate derivatives, credit derivatives and exotic derivatives
- Modeling the term structure of interest rates (Bootstrapping, short rate modelling) and credit spreads
- Credit scoring and provisioning
- Corporate financing activity prediction problems
- Portfolio optimization
- Real options
- Risk modeling (Financial risk modeling) and value at risk
- Dynamic financial analysis (DFA)

Financial models serve five purposes:

1. to demonstrate the size of the market opportunity
2. to explain the business model
3. to show the path to profitability
4. to quantify the investment requirement
5. to facilitate valuation of the business

CHAPTER TWO

FINANCIAL ALGEBRA

Function

It's the relationship between independent variable and the dependent variable. It consists of a constant and a variable.

A constant – This is a quantity whose value remains unchanged throughout a particular analysis e.g. fixed cost, rent, and salary.
A variable – This is a quantity which takes various values in a particular problem

Illustration

Suppose an item is sold at Sh 11 per unit. Let S represent sales rate revenue in shillings and let Q represents quantity sold.

Then the function representing these two variables is given as:

$S = 11Q$

S and Q are variables whereas the price - Sh 11 - is a constant.

Types of variables

Independent variable – this is a variable which determines the quantity or the value of some other variable referred to as the dependent variable. In Illustration 1.1, Q is the independent variable while S is the dependent variable.

An independent variable is also called a predictor variable while the dependent variable is also known as the response variable i.e. Q predicts S and S responds to Q.

3. A function – This is a relationship in which values of a dependent variable are determined by the values of one or more independent variables. In illustration 1.1 sales is a function of quantity, written as $S = f(Q)$

Demand = f(price, prices of substitutes and complements, income levels,….) Savings = f(investment, interest rates, income levels,….)

Note that the dependent variable is always one while the independent variable can be more than one.

Types of functions/equations

1) Linear equation

It takes the form **y = a + bx**

Where x and y are variables while a and b are constants.

e.g **y = 20 + 2x**

y = 5x

y = 15-0.3x

In graphical presentation of a linear equation, the constant 'a' represents y-intercept and 'b' represents the gradient of the slope. $\frac{\Delta y}{\Delta x}$

The linear equation can be presented either as 2 by 2 simultaneous equation. In general 2 by 2 equations are given as follows:

2x2

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

3x3 is given by as,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

2) Quadratic equations

The general formula is $ax^2 + bx + c = 0$

When the equation is ploted, it yields either a valley or a mountain depending on constant a. if < 0 a mountain, if> 0 a valley.

In order to solve a linear equation, the equation is equated to zero.

Methods of solving simultaneous equations:

Substitution Method
Example:

$$x = 86000 + 0.01y$$
$$y = 44000 + 0.02x$$

Rewrite:

$$86000 + 0.01y - x$$
$$44000 + 0.02x - y$$

$$Y = 44000 + 0.02(86000 + 0.01)$$
$$Y = 44000 + 1720 + 0.0002y$$

$$y - 0.0002y = 44000 + 1720$$

$$0.9998y = 45720$$

$$y = \frac{45720}{0.998}$$

$$= 45{,}729.15$$

$$X = 86000 + (0.01x45.729)$$

$$= 86457.29$$

Elimination

$$x + 2y = 3$$
$$2x + 3y = 4$$

$$2(x + 2y) = 3(2)$$

$$2x + 4y = 6$$
$$-2x + 3y = 4$$
$$y = 2$$

$$x = -3$$

Solving a 3× simultaneously equation

There are two main methods:

1. Matrix method
2. Crammers rule

1. Matrix method

A matrix is a rectangular array of numbers

Operations of matrices

The following operations can be carried out in matrices:
1. Addition
2. Subtraction
3. Multiplication
4. Determinant
5. Transposition
6. Matrix Inversion

Basics of Matrix Algebra

Definitions

A **matrix** is a collection of numbers ordered by rows and columns. It is customary to enclose the elements of a matrix in parentheses, brackets, or braces. For example, the following is a matrix:

$$\mathbf{X} = \begin{pmatrix} 3 & 8 & 6 \\ 2 & 1 & 5 \end{pmatrix}$$

This matrix has two rows and three columns, so it is referred to as a *2 by 3* matrix. The elements of a matrix are numbered in the following way:

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}$$

That is, the first subscript in a matrix refers to the row and the second subscript refers to the column. It is important to remember this convention when matrix algebra is performed.

A **vector** is a special type of matrix that has only one row (called a **row vector)** or one column (called a **column vector).** Below, **a** is a column vector while **b** is a row vector.

$$\mathbf{a} = \begin{pmatrix} 8 \\ 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 & 1 & 5 \end{pmatrix}$$

A **scalar** is a matrix with only one row and one column. It is customary to denote scalars by italicized, lower case letters (e.g., *x*), to denote vectors by bold, lower case letters (e.g., **x**), and to denote matrices with more than one row and one column by bold, upper case letters (e.g., **X**).***Special Names***

A **square matrix** has as many rows as it has columns. Matrix **A** is square but matrix **B** is not square:

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 12 & 5 \\ -1 & 7 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 12 & 5 \end{pmatrix}$$

A **symmetric matrix is** a square matrix in which $x_{ij} = x_{ji}$, for all *i* and *j*. Matrix **A** is symmetric; matrix **B** is not symmetric.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 12 & 10 \\ -1 & 10 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & -1 \\ 10 & 12 & 2 \\ -1 & 10 & 0 \end{pmatrix}$$

A **diagonal matrix** is a symmetric matrix where all the off diagonal elements are 0. These matrices are often denoted with **D**; matrix **D** is diagonal.

$$\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

An **identity matrix** is a diagonal matrix with 1s and only 1s on the diagonal. The identity matrix is almost always denoted as **I**.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix Algebra

Addition and Subtraction

To add two matrices, they both must have the same number of rows and they both must have the same number of columns. The elements of the two matrices are simply added together, element by element, to produce the results. That is, for **R** = **A** + **B**, then $r_{ij} = a_{ij} + b_{ij}$.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 12 & 10 \\ -1 & 10 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 9 \\ 3 & 10 & 5 \\ -1 & 6 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -10 \\ -1 & 2 & 5 \\ 0 & 4 & 0 \end{pmatrix}$$

Matrix subtraction works in the same way, except that elements are subtracted instead of added.

Matrix Multiplication

There are several rules for matrix multiplication. The first concerns the multiplication between a matrix and a scalar. Here, each element in the product matrix is simply the scalar multiplied by the element in the matrix. That is, for **R** = a**B**, then $r_{ij} = ab_{ij}$ for all i and j. Thus,

$$4\begin{pmatrix} 3 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 12 & 8 \\ 12 & 24 \end{pmatrix}$$

Matrix multiplication involving a scalar is commutative. That is, $a\mathbf{B} = \mathbf{B}a$. The next rule involves the multiplication of a row vector by a column vector. To perform this, the row vector must have as many columns as the column vector has rows.

For example, $\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is legal.

But, $\begin{pmatrix} 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is not legal because the row vector has three columns while the column vector has two rows.

The product of a row vector multiplied by a column vector will be a scalar. This scalar is simply the sum of the first row vector element multiplied by the first column vector element plus the second row vector element multiplied by the second column vector element plus the product of the third elements, etc. In algebra, if $r = \mathbf{ab}$, then $r = \sum_{i=1}^{n} a_i b_i$. Thus,

$$\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0*0 + 1*1 + 2*2 = 5$$

All other types of matrix multiplication involve the multiplication of a row vector and a column vector. Specifically, in the expression $\mathbf{R} = \mathbf{AB}$, $r_{ij} = \mathbf{a}_{i\bullet}\mathbf{b}_{\bullet j}$ where $\mathbf{a}_{i\bullet}$ the ith row vector in matrix **A** and $\mathbf{b}_{\bullet j}$ is the jth column vector in matrix **B**. Thus, if

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 2 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 7 & -1 \end{pmatrix}$$

then

$$r_{11} = a_{1\bullet}b_{\bullet 1} = \begin{pmatrix} 1 & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} = 1*1 + 5*0 + 1*7 = 8$$

and

$$r_{12} = a_{1\bullet}b_{\bullet 2} = \begin{pmatrix} 1 & 5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 1*2 + 5*4 + 1*-1 = 19$$

and

$$r_{21} = a_{2\bullet}b_{\bullet 1} = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} = 0*1 + 2*0 + 1*7 = 7$$

and

$$r_{22} = a_{2\bullet}b_{\bullet 2} = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 0*2 + 2*4 + 1*-1 = 7$$

Hence,

$$\mathbf{AB} = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 7 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 19 \\ 7 & 7 \end{pmatrix}$$

For matrix multiplication to be legal, the first matrix must have as many columns as the second matrix has rows. This, of course, is the requirement for multiplying a row vector by a column vector. The resulting matrix will have as many rows as the first matrix and as many columns as the second matrix. Because **A** has 2 rows and 3 columns while **B** has 3 rows and 2 columns, the matrix multiplication may legally proceed and the resulting matrix will have 2 rows and 2 columns.

Because of these requirements, matrix multiplication is usually not commutative. That is, usually $\mathbf{AB} \neq \mathbf{BA}$. And even if **AB** is a legal operation, there is no guarantee that **BA** will also be legal. For these reasons, the terms premultiply and postmultiply are often encountered in matrix algebra while they are seldom encountered in scalar algebra.

One special case to be aware of is when a column vector is postmultiplied by a row vector. That is, what is

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\begin{pmatrix} 2 & 4 \end{pmatrix}?$$

In this case, one simply follows the rules given above for the multiplication of two matrices. Note that the first matrix has one column and the second matrix has one row, so the matrix multiplication is legal. The resulting matrix will have as many rows as the first matrix (3) and as many columns as the second matrix (2). Hence, the result is

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 6 & 12 \\ 8 & 16 \end{pmatrix}$$

Similarly, multiplication of a matrix times a vector (or a vector times a matrix) will also conform to the multiplication of two matrices. For example,

$$\begin{pmatrix} 4 & 8 \\ 6 & 12 \\ 8 & 16 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

is an illegal operation because the number of columns in the first matrix (2) does not match the number of rows in the second matrix (3). However,

$$\begin{pmatrix} 4 & 8 \\ 6 & 12 \\ 8 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4*1+8*0 \\ 6*1+12*0 \\ 8*1+16*0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 5 & 5 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 0*3+2*5+3*7 & 0*7+2*5+3*3 \end{pmatrix} = \begin{pmatrix} 31 & 19 \end{pmatrix}$$

The last special case of matrix multiplication involves the identity matrix, **I**. The identity matrix operates as the number 1 does in scalar algebra. That is, any vector or matrix multiplied by an identity matrix is simply the original vector or matrix. Hence, **aI** = **a**, **IX** = **X**, etc. Note, however, that a scalar multiplied by an identify matrix becomes a diagonal matrix with the scalars on the diagonal. That is,

$$2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and not 2. This should be verified by reviewing the rules for multiplying a scalar and a matrix given above.

Matrix Division

For simple numbers, division can be reduced to multiplication by the reciprocal of the divider: 32 divided by 4 is the same as 32 multiplied by 1/4 , or multiplied by 4^{-1}, where 4^{-1} is defined by the general equality a^{-1} a = 1.

When working with matrices, we shall adopt the latter idea, and therefore not use the term division at all; instead we take the multiplication by an inverse matrix as the equivalent of division. However, the computation of the **inverse matrix** is quite complex, and discussed shortly.

Matrix Transpose

The transpose of a matrix is denoted by a prime (**A'**) or a superscript t or T ($\mathbf{A}^t$ or $\mathbf{A}^T$). The first row of a matrix becomes the first column of the transpose matrix, the second row of the matrix becomes the second column of the transpose, etc. Thus,

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 2 & 1 \end{pmatrix} \qquad \mathbf{A}^t = \begin{pmatrix} 1 & 0 \\ 5 & 2 \\ 2 & 1 \end{pmatrix}$$

The transpose of a row vector will be a column vector, and the transpose of a column vector will be a row vector. The transpose of a symmetric matrix is simply the original matrix.

Matrix Inverse

In scalar algebra, the inverse of a number is that number which, when multiplied by the original number, gives a product of 1. Hence, the inverse of x is simple 1/x. or, in slightly different notation, x^{-1}. In matrix algebra, the inverse of a matrix is that matrix which, when multiplied by the original matrix, gives an identity matrix. The inverse of a matrix is denoted by the superscript "-1". Hence,

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

A matrix must be square to have an inverse, but not all square matrices have an inverse. In some cases, when the **determinant** of the matrix is zero, the inverse does not exist. For covariance and correlation matrices, an inverse will always exist, provided that there are more subjects than there are variables and that every variable has a variance greater than 0.

It is important to know what an inverse is in multivariate statistics, but it is not necessary to know how to compute an inverse. There are several ways to compute the inverse. The general way is by solving a set of structural equations in which the elements of the inverse matrix are the unknowns. A simple way to compute the inverse, involves the transpose, but this only works when the column vectors are **orthogonal**, that is when $\mathbf{x}_{\bullet 1} \bullet \mathbf{x}_{\bullet 2} = 0$. When

$\mathbf{A} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is orthogonal, then $\begin{aligned} x_{11}x_{12} + x_{21}x_{22} &= 0 \\ x_{11}^2 x_{12}^2 &= 1 \\ x_{21}^2 x_{22}^2 &= 1 \end{aligned}$ will hold.

When matrix **A** is transposed we obtain $\mathbf{A}^t$.

$$\mathbf{A}^t = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$$

When we compute $\mathbf{A}^t \mathbf{A}$ we obtain:

$$\mathbf{A}^t\mathbf{A} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}^2 x_{12}^2 & x_{11}x_{12} + x_{21}x_{22} \\ x_{11}x_{12} + x_{21}x_{22} & x_{21}^2 x_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Therefore, when **A** orthogonal, then $\mathbf{A}^t$ is $\mathbf{A}^{-1}$.

The inverse of a product of two matrices is the swapped product of the individual inverse matrices. Thus: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. Where it is assumed that **A** and **B** are square and that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist. The proof is straightforward. Let **AB** be given, then we have

$$(\mathbf{AB})(\mathbf{AB})^{-1} = (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1} = \mathbf{A}\mathbf{I}\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$

Determinant of a Matrix

The determinant of a matrix is a scalar and is denoted as $|\mathbf{A}|$ or det(**A**). The determinant is a value. It has very important mathematical properties, but it is very difficult to provide a substantive definition. It requires some steps to show how the value is found.

If **A** is of order *n* x *n*, then the determinant is said to be or order *n*. Given a determinant |**A**| of order *n*, we can form products of *n* elements in such a way that from each row and column of |**A**| one and only one element is selected as a factor for the product. This is more easily seen in an example. Suppose |**A**| is of third order:

$$\mathbf{A} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \text{ then } \quad |\mathbf{A}| = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}$$

Then the product $x_{12}x_{23}x_{31}$ would satisfy the requirement: we have x_{12} as the only element form the first row (second column), x_{23} as the only element from the second row (third column), and x_{31} as the only element form the third row (first column). Such a product is called a *term* of the determinant

However, we can form many terms like this. Other examples are $x_{12}x_{21}x_{33}$, $x_{13}x_{22}x_{31}$. The complete set appears to be:

$x_{11}x_{22}x_{33}$, $x_{11}x_{32}x_{23}$

$x_{12}x_{21}x_{33}$, $x_{12}x_{23}x_{31}$

$x_{13}x_{21}x_{32}$, $x_{13}x_{22}x_{31}$

Note that there is systematicity in this. Start with the first row and than multiply with the left and right diagonal elements of the submatrices. Or more formal, because we have one element out of each row, we can always rank the elements in each term in such a way the row indices follow the natural numbers. All that is left is to determine an order for the second subscripts; clearly they can be taken in as many ways as there are permutations, we can form 3! = 6 terms for a determinant of order 3. In general, a determinant of order *n* will have *n*! terms.

The next step is to assign a plus or a minus sign to each term. Assuming again that the row subscripts are in natural order, the sign depends on the column subscripts only. First we shall agree that every time a higher subscript precedes a lower, we have an inversion. Looking back at the terms just presented, we have 0, 1, 1, 2, 2, and 3 inversions. For terms with an odd number of inversions, the sign becomes negative.

It is now possible to compute the determinant of **A**.

$$|\mathbf{A}| = x_{11}x_{22}x_{33} - x_{11}x_{32}x_{23} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}$$

A numerical example:

The following rules are important for, and can help you sometimes:

1. The determinant of A has the same value as the determinant of A'.
2. The value of the determinant changes sign if one row (column) is interchanged with another row (column).
3. If a determinant has two equal rows (columns), its value is zero.
4. If a determinant has two rows (columns) with proportional elements, its value is zero.
5. If all elements in a row (column) are multiplied by a constant, the value of the determinant is multiplied by that constant.
6. If a determinant has a row (column) in which all elements are zero, the value of the determinant is zero.
7. The value of the determinant remains unchanged if one row (column) is added to or subtracted form another row (column). Moreover, if a row (column) is multiplied by a constant and then added to or subtracted from another row (column) the value remains unchanged.

For covariance and correlation matrices, the determinant is a number that is sometimes used to express the "generalized variance" of the matrix. That is, covariance matrices with small determinants denote variables that are redundant or highly correlated (this is something that is used in factor analysis, or regression analysis). Matrices with large determinants denote variables that are independent of one another. The determinant has several very important properties for some multivariate stats (e.g., change in R^2 in multiple regression can be expressed as a ratio of determinants). It is obvious that the computation of the determinant is a tedious business, so only fools calculate the determinant of a large matrix by hand. We will try to avoid that, and have the computer do it for us.

Illustration 1:
Jane Mary and Joseph purchased cereals A, Band C from Faida super market. Jane purchased 1kg of A, 3kg of B and C and spent a total of ksh.650 on each Mary purchased 1kg, 4kg and 2kg of A, B, C and spent ksh.700. Joseph bought 2kg of A&B and 1kg of C and spent 650

I. Express the following information in a matrix form
II. Determine unit price of each type of cereals

solution

Let X -Jane
Y -mary
Z -joseph

$$\begin{bmatrix} A+3B+3C \\ A+4B+2C \\ 2A+2B+C \end{bmatrix} = \begin{bmatrix} 650 \\ 700 \\ 650 \end{bmatrix}|$$

$$\begin{bmatrix} 1+3+3 \\ 1+4+2 \\ 2+2+1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 650 \\ 700 \\ 650 \end{bmatrix}$$

$Det = 1(4x1 - 2x2) - 3(1x1 - 2x2) + 3(1x2 - 2x4)$

$Det = (1x0) - (3x3) + (3x - 6)$

$= 0 + 9 - 18$

$= -9$

Inverse method

To find an inverse of a matrix, find

- Determinant
- Minor
- Co-factor
- Adjoint
- Divide adjoint with determinant

$$\begin{bmatrix} 0 & -3 & -6 \\ -3 & -5 & -4 \\ -6 & -1 & 1 \end{bmatrix}$$

Row 2 (3x1)-(2x3)= -3
(1x1)-(2x3)= -5
(1x2)-(2x3)= -4

Row 3 (3x2)-(4x3)= -6
(1x2)-(1x3)= -1
(1x4)-(1x3)= -1

Cofactor

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{matrix} 0 & 3 & 6 \\ 3 & -5 & 4 \\ -6 & 1 & 1 \end{matrix}\Bigg]$$

Adjoint

$$\begin{bmatrix} 0 & 3 & 6 \\ 3 & -5 & 4 \\ -6 & 1 & 1 \end{bmatrix} \frac{1}{-9}$$

$$A^{-} = |\mathrm{A}| \text{ x } \mathrm{A}^{\mathrm{T}}$$

$$A^{-} = \begin{matrix} 0 & {}^{-3}/_{9} & {}^{6}/_{3} \\ {}^{-3}/_{9} & {}^{5}/_{9} & {}^{-1}/_{9} \\ {}^{6}/_{9} & {}^{-4}/_{9} & {}^{-1}/_{9} \end{matrix}\Bigg]$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{matrix} 0 & {}^{-3}/_{9} & {}^{6}/_{3} \\ {}^{-3}/_{9} & {}^{5}/_{9} & {}^{-1}/_{9} \\ {}^{6}/_{9} & {}^{-4}/_{9} & {}^{-1}/_{9} \end{matrix}\Bigg] \begin{bmatrix} 650 \\ 700 \\ 650 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{matrix} 0\text{x}650 & {}^{-3}/_{9}\,\text{x}700 & {}^{6}/_{3}\,\text{x}650 \\ {}^{-3}/_{9}\,\text{x}650 & {}^{5}/_{9}\,\text{x}700 & {}^{-1}\ {}_{9}\,\text{x}650 \\ {}^{6}/_{9}\,\text{x}650 & {}^{-4}/_{9}\,\text{x}700 & {}^{-1}\ {}_{9}\,\text{x}650 \end{matrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \\ 50 \end{bmatrix}$$

CRAMMER'S METHOD

A:

$$\frac{\begin{vmatrix} \boldsymbol{h_1} & b & c \\ \boldsymbol{h_2} & e & f \\ \boldsymbol{h_3} & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = X_1$$

$$\frac{\begin{vmatrix} 650 & 3 & 3 \\ 700 & 4 & 2 \\ 650 & 2 & 1 \end{vmatrix}}{-9} = A$$

Numerator $650[(4x1) - (2x2)] - 3[(700x1) - (650x3)] + 3[(700x2) - (650x4)]$

$$= -1800$$

$$A = \frac{-1800}{-9}$$

$$A = 200$$

B:

$$\frac{\begin{vmatrix} a & \boldsymbol{h_1} & c \\ d & \boldsymbol{h_2} & f \\ g & \boldsymbol{h_3} & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = X_2$$

$$\frac{\begin{vmatrix} 1 & 650 & 3 \\ 1 & 700 & 2 \\ 2 & 650 & 1 \end{vmatrix}}{-9} = B$$

$$B = \frac{-900}{-9}$$

$$B = 100$$

C:

$$\frac{\begin{vmatrix} 1 & 3 & 650 \\ 1 & 4 & 700 \\ 2 & 2 & 650 \end{vmatrix}}{-9} = C$$

$$C = \frac{-450}{-9}$$

$$C = 100$$

<u>REDUCTION METHOD</u>

$$A + 3B + 3C = 650$$
$$A + 4B + 2C = 700$$
$$2A + 2B + C = 650$$

$$A + 3B + 3C = 650$$
$$\underline{A + 4B + 2C = 700}$$
$$-B + C = -50 \ldots \ldots . . (i)$$

$$A + 4B + 2C = 700 \ldots \times 2$$
$$2A + 2B + C = 650$$
$$-B + C = -50 \ldots \ldots . . (i)$$

$$2A + 8B + 4C = 1400$$
$$\underline{2A + 2B + C = 650}$$
$$6B + 3C = 750 \ldots \ldots . . (ii)$$

$$-B + C = -50$$
$$6B + 3C = 750$$

Make B the subject

$$B = 50 + C$$
$$= 6(50 + C) + 3C = 750$$
$$= 300 + 6C + 3C = 750$$
$$9C = 750 - 300 = 450$$

$$C = {}^{450}/_{9} = 50$$
$$6B + 3C = 750$$
$$6B + (3 \times 50) = 750$$
$$6B = 750 - 150 = 600$$
$$B = \frac{600}{6} = 100$$

Thus,

$$A + 3B + 3C = 650$$
$$A + (3 \times 100) + (3 \times 50) = 650$$
$$A = 650 - 300 - 150$$
$$A = 200$$

Illustration 2:

Mary is employed as a sales lady in a company she is paid a basic salary and a commission on sales paid. Mary is paid a commission of sh.x on additional sales made above sh.100,000. During the month of July, august and September 2004: Mary made the sales and gross earnings as shown:

Months	Sales	Gross earning
July	300 000	23 000
August	200 000	20 000
September	50 000	14 750

In September 2004, she had 2weeks sick leave thus sales were low.

a) Rates and commission applied.
b) Basic salary
c) Gross earnings of October 2004 when she made sales worth sh.400, 000.

Solution:

a) Rates and commission applied.

$$= \frac{100\ 000}{100} x 1000x$$

$$= 1000x + \left(\frac{33\ 000 - 100\ 000}{100}\right)y + B = 23\ 000$$

$$= 1000x + 2000y + B = 23\ 000 \dots\dots i$$

$$= 1000x + \left(\frac{200\ 000 - 100\ 000}{100}\right)y + B = 20\ 000$$

$$= 1000x + 1000y + B = 20\ 000 \dots\dots ii$$

$$\begin{array}{r} 1000x + 2000y + B = 23\ 000 \\ \underline{1000x + 1000y + B = 20\ 000} \\ 1000y = 3\ 000 \end{array}$$

$$y = \frac{3000}{1000} = 3 \qquad (3\%)$$

$$= B + \left(\frac{50\ 000}{100}\right)x = 14\ 750$$

$$\begin{array}{r} B + 500x = 14\ 750 \dots\dots\dots iii \\ \underline{-B + 1000x = 17\ 000} \\ -500x = -2250 \end{array}$$

$$x = \frac{-2250}{-500} = 4.5 \qquad (4.5\%)$$

b) Basic salary

$$B + 500x = 14750$$
$$B + (500 \times 4.5) = 14750$$
$$B = 14750 - 2250$$

$$B = 12{,}500 \qquad G = a + bx$$

c) Gross earnings of October 2004 when she made sales worth sh.400, 000

$$B + X + Y$$

$$12500 + \left(\frac{400000}{100}\right)5 + \left(\frac{300000}{100}\right)3$$

$$= 12500 + 4500 + 9000$$

$$= 26000$$

<u>Application of equation or functions in cost-volume-profit analysis</u>

It can be applied in the following areas:

a) Demand function (price function)